European Journal of Science and Theology, June 2022, Vol.18, No.3, 123-148

THE 'GEOMETRIC MANIFESTO' OF FRAY IGNACIO MUÑOZ (1684)
THE KEPLER HERESY AND THE HEPTAGONAL APSES OF GOTHIC CATHEDRALS

Josep Lluis i Ginovart* **and Cinta Lluis-Teruel**

Universitat Internacional de Catalunya, School of Architecture Barcelona,
C/ Immaculada 22, Barcelona, 08017, Spain

(Received 10 January 2022, revised 30 March 2022)

Abstract

Johannes Kepler raised the problem of the incommensurability of the construction of the heptagon in the 'Harmonices mundi libri' V, (1619). Thus, he stated that the entities susceptible of knowledge, and how such a figure, whose formal description is impossible, are not susceptible to that knowledge. Therefore it cannot be known by the human mind, being beyond the finite that the Creator constructs. The work had an inquisitorial response in the 'Manifiesto Geometrico' (1684) by the Dominican Ignacio Muñoz, dedicated to the construction of the heptagon through the isosceles triangle (9,9,4). The Dominican friar died without knowing that his method, using a commensurable ratio (9:4), similar to that on the 'geometria fabrorum' of the Gothic architects in the heptagonal apses, it would be one of the precise methods that practical geometries have developed up to the 21st century.

Keywords: mathematics, 17th century, geometry, heptagon, fray Ignacio Muñoz

1. The unknown heptagon of the 'Harmonices mundi'

Johannes Kepler (1571-1630), a Protestant scientific, raised the problem of the incommensurability of the construction of the heptagon in the *Harmonices mundi libri V* [1]. It was after explaining to his Catholic friend Hans Georg Herwart von Hohenburg (1553-1622) that the celestial machine was not created as a divine animal, but as a clock governed by a force that can be expressed mathematically [2]. He explains it from his acquaintance with the Swiss watchmaker and mathematician Jost Bürgi (1552-1632), who moved to Prague in 1603 and is quoted in the *XLV. Propositio* dedicated to the construction of the heptagon [3]. Quite a revelation after his first astronomical work *Prodromus dissertationum cosmographicarum, continens mysterium cosmographicum*

*E-mail: jlluis@uic.es

(1596), *(The Sacred Mystery of the Cosmos)*, in which the construction of the Universe was ordered with the expansion of the regular polyhedra of Platonic base, cube, tetrahedron, dodecahedron, icosahedron and octahedron. It was provided in a numerical origin of integer number base [4] (Figure 1a), and it will be the cosmological ordering of the regular figures of the Caput 1, Lib. V of the *Harmonices mundi* [1, p. 180-182] (Figure 1b).

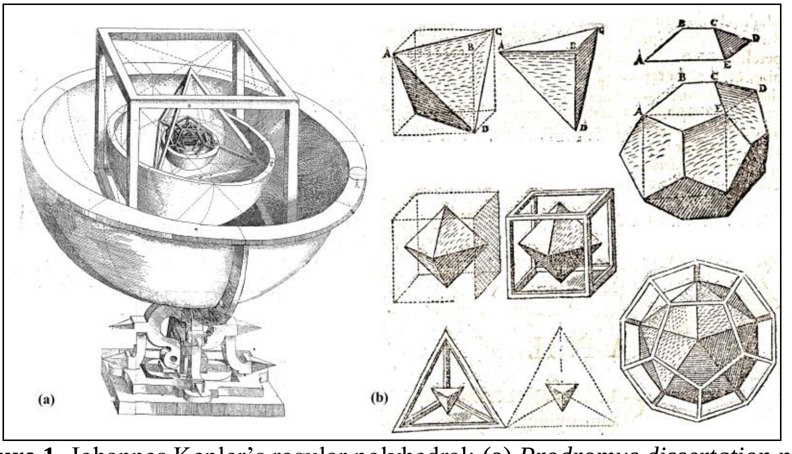

Figure 1. Johannes Kepler's regular polyhedral: (a) *Prodromus dissertation num cosmographicarum* (1596) [4, Tabula III, https://www.e-rara.ch/doi/10.3931/e-rara-445], (b) *Keppleri Harmonices mundi libri V* (1619) [1, p. 181, https://www.e-rara.ch/zut/content/titleinfo/2434556]. Detail: Terms of Use, Licence, Public Domain Mark

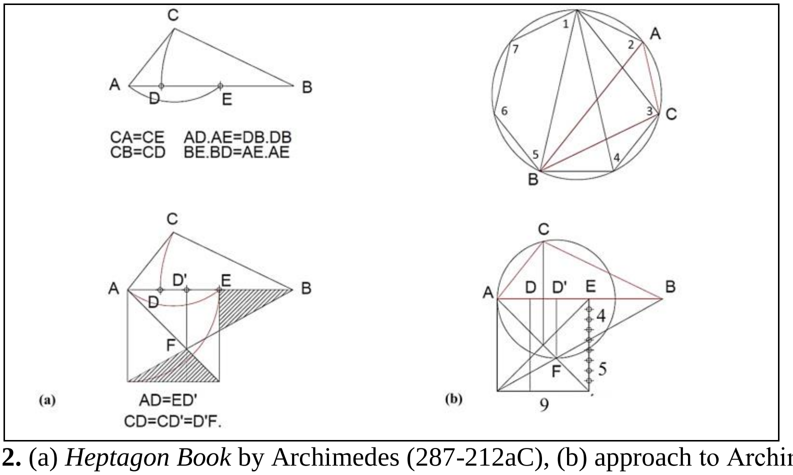

Figure 2. (a) *Heptagon Book* by Archimedes (287-212aC), (b) approach to Archimedes' method. Author's figure.

Kepler deals with the analysis of the heptagon from the decomposition of the five congruent inner triangles that can be formed with the seven vertices. Two of them are opposite to the third, according to the tradition of the *Heptagon Book by Archimedes* (287-212 BC) (Figure 2a), in which he approaches the figure from purely mathematical aspects [5] (Figure 2b). This knowledge was transmitted by Abu Ali al-Hasan ibn al-Haytham (c.965 - c.1040) [6], with two ways of dealing with the study of the figure. On one hand from the proportional division of the segment into three parts by Abu Sahl Waijan ibn Rustam al-Quhi (c.940 - c.1000) and on the other hand through the trisection of the angle by Abu Said Ahmad ibn Muhammad Al-Sijzi (c.945 - c.1020) [7].

Kepler knew well the reference of the heptagon by Cristoforo Clavio (1538-1612) from *the Geometria practica* (1604) Theor 12. Propos. 30. He analysed the tracing outlines of Albert Dürer (1471-1528), Carolus Marianus Cremonensis (f.1599) (Figure 3a) and François de Foix de Candale (1502-1594) [8]. In addition with Jost Bürgi (1552-1632) and Pier Francesco Malaspina (1550-1624) (Figure 3b) [9]. He states that the figure could not have been constructed consciously, nor could it have been made by the methods used so far. Therefore it cannot be confirmed if they really could have done it or whether they did it by chance. He will replicate Girolamo Cardano's solution of the heptagon as set out in the Nuremberg editions of *Subtilitate Libri XXI* [10] and extended in the Lyon edition [11]. He considers the construction of the heptagon by the inner scalene triangles with the *proportio reflexa* [12]. Kepler also warns that it is not necessary to be a geometrical expert to see that Dürer's proposal is mistaken when he suggests taking the side of the heptagon equal to half the side of the equilateral triangle inscribed in a circle as an approximation of the square root $\sqrt[2]{3/2}$ of the side of the heptagon with respect to the radius of the circle (Figure 4b) [13].

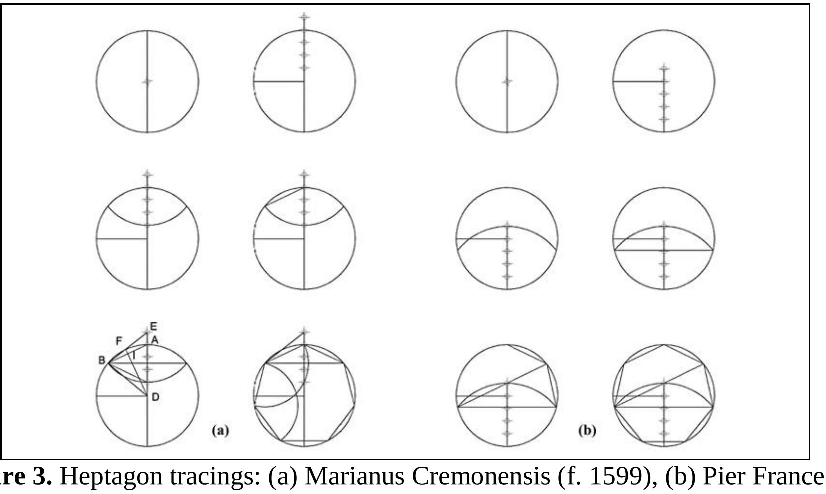

Figure 3. Heptagon tracings: (a) Marianus Cremonensis (f. 1599), (b) Pier Francesco Malaspina (1550-1624). Author's figure.

The research deals with the criticism made by the Dominican friar Ignacio Muñoz Pinciano (c.1608-1685) in the *Manifiesto geometrico* (1684) [14], who claims to have discovered a method of tracing the heptagon, against the development of the figure determined by Kepler whom he accuses of being a heretic. The Dominican will construct the figure through the isosceles triangle (9,4,9) using the commensurable relation (9:4) between the side of the heptagon and its diagonal, similar to that of the *geometria fabrorum* of the layout of the heptagonal apses of the Gothic architects. The author belongs to the group of second order mathematicians in the reign of Charles II (1665-1700) led by Father José de Zaragoza (1627-1679), Juan Caramuel Lobkowitz (1606-1682) and Antonio Hugo de Omerique (1634-1705) [15].

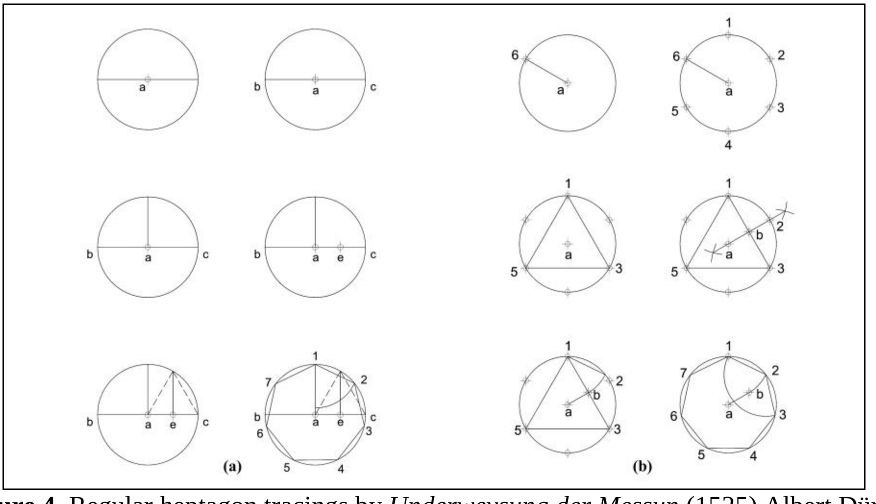

Figure 4. Regular heptagon tracings by *Underweysung der Messun* (1525) Albert Dürer: (a) pentagon tracing (LII.15) [16, fol. 27 r], (b) heptagon tracing (LII.11) [16, fol. 26 r]. Author's figure.

2. The practical constructions of the heptagon

The most widely used practical construction of the heptagon that has come down to us determines the side of the regular heptagon as the height of the equilateral triangle of side equal to the radius inscribed in the circumference of Albert Dürer's *Underweysung der Messung*. It is a consequence of the corollary of the pentagon tracing (LII.15) (Figura 4a), in addition to Kepler's own method of the heptagon figure (LII.11) criticized by Kepler (Figura 4b) [16].

This method was explained (Inst. 25, fig. 9) in *L'architettura civile, preparata su la geometria e ridotta alle prospettive* by F. Galli-Bibiena (1657-1743) [17] and Matila C. Ghyka's (1881-1965) *Esthétique des Proportions dans la Nature et dans les Arts* (1927) [18]. The origin of these practical layouts can be traced back to the *Kitāb fī mā yaḥtāju al-ṣāni' min al-a'māl al-handasiyya* (*Book on those geometrical constructions which are necessary for craftsmen*) (c.993-1008), by Mohammad Abu'l-Wafa Al-Buzjani (940-998) (Figure 5a) [19]. It arrived to the Latin West through Ibn Yūnus, Ka māl al-Dīn (1156-1242) with the Sharh: *a'māl al-handasiyya li Abū al-Wafā* (1240), (*Commentary on Geometric Constructions by Abu'l-Wafa'*) at the court of Emperor Frederick II (1194-1250) [20]. The dissemination was done through *Geometria Deutsch* (1472) attributed to Hans Hösch von Gmünd (f. 1472) [21] and the *Geometrie Deutsch* (1488) by Matthäus Roriczer (+c. 1495) (Figure 5b) [22]. While the trisection of the angle will be developed in the Latin West by Jordanus Nemonarius (1225-1260) in the *Geometria vel de triangulis libri IV*, (Liber IV, 23) [23], being the link with the classical tradition with *Varii de Rebus Mathematicis Responsi, Liber VIII* (1593) by François Viète (1540-1603) and his Protasis IV Theorema [24].

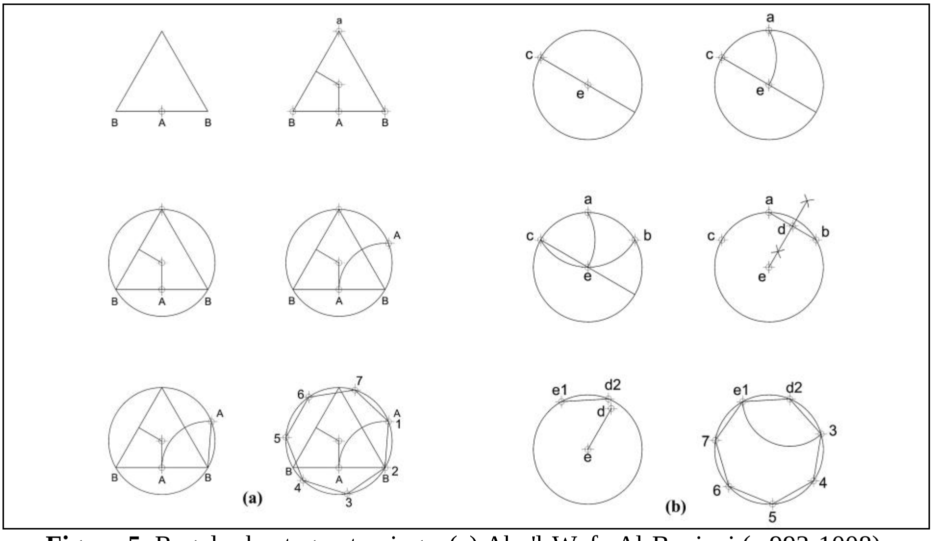

Figure 5. Regular heptagon tracings: (a) Abu'l-Wafa Al-Buzjani (c.993-1008), (b) Matthäus Roriczer (1488). Author's figure.

Another possible classical connection to the construction of the heptagon is based on the proportion between the side of the polygon inscribed in the surface of a heptagon and its diameter, which is the case of the pseudo-Heronian *Metrica*, attributed by Hero of Alexandria (c.20-62) (I, 29 Theorem 54), *Dimetiendi rationes* (I, XX). Starting from the regular hexagon, Heronis deduces that the equilateral triangle constructed by using the radius, to which he assigns a length of 8 units, it is 7 units high. Hence in proposition a heptagon of side 7 has a radius of 8, providing the proportion with the circumference diameter of 16:7 [25]. Other approaches can be found in *Pseudogeometría Geometry II* by Boethius defined in *De multiangulis figuris*, *De eptagoni* [26]. Similarly, an approach can be found in the work of Giorgio Valla (1447-1501), *De expetendis et fugiendis rebús*, with one part dedicated to the six books of Geometry [27].

Kepler claims to deal with entities susceptible of knowledge, and the heptagon is not among these entities, since its formal description it is impossible, and therefore it cannot be known by the human mind. The possibility of constructing the figure, it is a priori, the possibility of being able to be known; for this reason, it cannot be known by the omniscient mind as a simple eternal act either, since, he says, its nature is amongst the unknowable things.

3. The 'Manifiesto geometrico', fray Ignacio Muñoz and the apology against Kepler

The Dominican Ignacio Muñoz had a troubled scientific life in Manila where he arrived in 1635 until his return in 1665 via New Spain, with a stopover in Mexico (1665-1669). He arrived at the Court of Charles II in 1670, publishing in 1684 the *Manifiesto geometrico, plus ultra de la geometria practica* in Brussels by Francisco Foppens (c.1600-1685) in 1684. The work was finished in 1678 and sent in 1683 to the Duke of Béjar and Plasencia, Manuel López de Zúñiga (1657-1686) to whom he dedicated the work and who acted as a patron (Figure 6a).

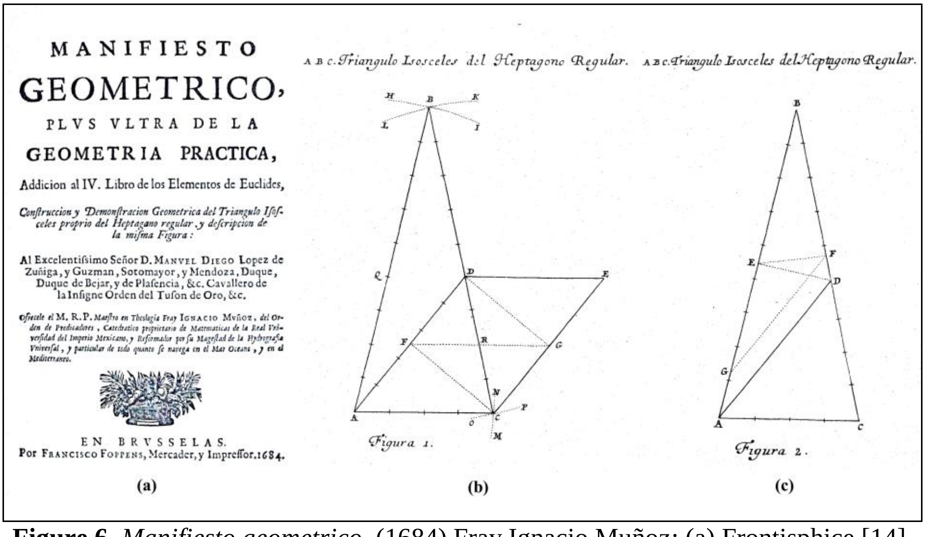

Figure 6. *Manifiesto geometrico*, (1684) Fray Ignacio Muñoz: (a) Frontisphice [14], (b) isosceles triangle inscribed in the heptagon [14, fig. 1], (c) isosceles triangle of the heptagon [14, fig. 2] [Biblioteca Nacional de España, Madrid (BNE), Sig. 3/48498, https:c//thales.cica.es/rd/Recursos/rd97/Otros/01_1_b.html] Detail: Terms of Use, Licence, Public Domain Mark.

In the title of the work the *Very Reverend Father Padre Fray Ignacio Muñoz*, he adds *Master of Theology, of the Order of Preachers*, a position he held during his stay in Manila (1635-1665) at the College of Saint Thomas in Goa [28]. Also he obtained the position of 'Full Professor of Mathematics at the Royal University of the Mexican Empire', during his stay in Mexico (1665-1669) after the death of the Mexican Mercedarian religious Diego Rodríguez (1596-1668) [29]. The professorship had been created in 1637, and he held it until 1672, on his return to Spain. Later the creole Luis Becerra Tanco (c. 1602-1672) was appointed to the post but he held it only for three months, then it was taken over by the Mexican Jesuit, Carlos Sigüenza Góngora (1645-1700) [30]. Finally Fray Ignacio Muñoz was established as *Reformer by His Majesty of Universal Hydrography and in particular of everything that is navigated in the*

Ocean Sea and the Mediterranean, a title granted by the Court in October 1670. This position was presented to the Council of the Indies in July 1670, and it was endorsed by Father José de Zaragoza [28].

In the dedication of the *Manifesto*, fray Ignacio explains the request that the Duke of Béjar and Plasencia asked the Portuguese military Count of the Torre, Juan Mascareñas (1633-1681). He requested an expert opinion on the problem of the isosceles triangle of the heptagon from the Major Cosmographer of Portugal, Luis Serrán Pimentel (1613-1679). In the reply in 1677, before being finished the Dominican's work, he raised the difficulties of the solutions for the construction of the heptagon, naming the proposed by Carolus Marianus Cremonensis, François de Foix de Candale, Oroncio Fineo (1494-1555), Pedro Nunes (1502-1578), Johannes Kepler, François Viète and José Zaragoza y Vilanova. The difficulty of the resolution of the problem faced by Fray Ignacio Muñoz is expressed by Pimentel, who requires: "*If this Gentleman has found them, it will be something for which great glory will result for them*", a quote used by the Dominican in his dedication of the *Manifesto*.

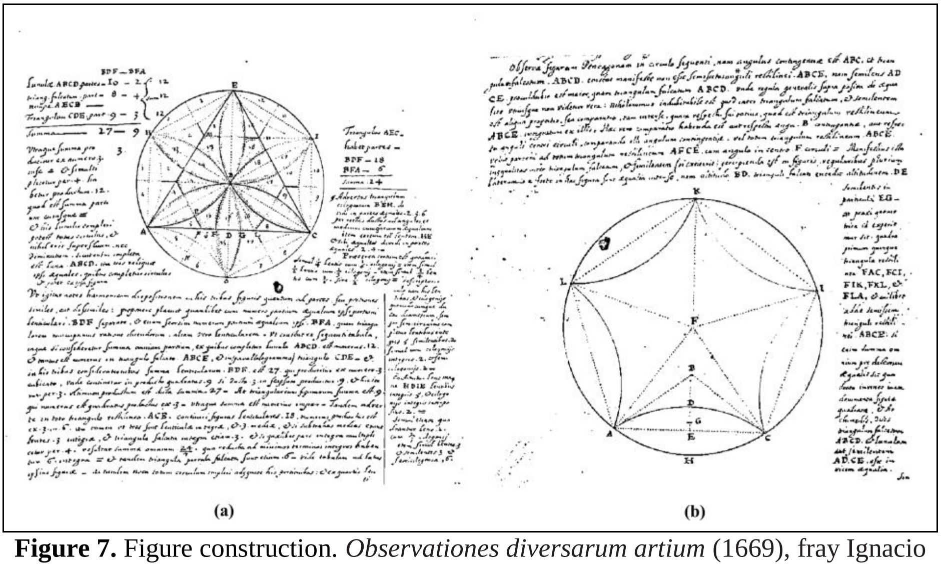

Figure 7. Figure construction. *Observationes diversarum artium* (1669), fray Ignacio Muñoz [BNE Mss/7111]: (a) hexagon (fol. 31), (b) pentagon (fol. 32). Detail: Terms of Use, Licence, Public Domain Mark.

Other manuscripts by the Dominican friar are known, such as: the *Observationes diversarum artium* (1669) [BNE, Mss/7111] (Figure 7); *Directions of the seas of Morocco, the Canary Islands, America and the Philippines, and other documents* (1669-1686) [BNE Mss/7119]; *The geometric synóptic and universal operation to divide any rectilinear angle into the equal or proportional parts that are requested* (1670) [Manuscrito Real Academia de la Historia, Madrid (M-RAH), 9/2782]; *The Communication sent to the Jesuit José Zaragoza, following a past discussion on the work of Gregoire de Saint-Vincent in which he will try to demonstrate the squaring of the circle* [M-RAH, 9/3638] and *Memorial and hydrographic manifest in which it is shown that all of Rio de la Plata and its island of San Gabriel and all the other islands and lands that bathe this mighty river are and belong to the conquests and domain of the Crown of Castile* [M-RAH, 9/2810].

Other documents have disappeared, such as the *Demostratio Geometrica trianguli Ysosceles, in Heptagono regulari*; *Geometría práctica*; *Novus Geometricae thesaurus*; *Hydrographia Universalis et particularis practica et speculativa*; *Descriptio currum siam*; *Architectura communis*, and *Tabula declinationis solis et stellarum* [31].

Finally there are some unfinished works such as *Nuevo Tesoro, y Plus Ultra de la Geometria Practica* in which he wanted to deal with odd-sided figures; 9-11-13-15-17 and 19 [14, p. 29]. He also draws the first planimetry of Manila, *Descripción geométrica de la ciudad y circunvalación de Manila y de sus arrabales al Consejo de las Indias. Por el Padre Maestro Fray Ignacio Muñoz, del Orden de Predicadores. Año 1671* [Archivo General Indias, Sevilla] [32].

The *Manifesto* is divided into two different parts, the first dedicated to the construction of the heptagon as an addition to Euclid's *Elementa* [14, p. 1-19], with the isosceles triangle proper to the septangle (9,4,9). He uses a commensurable arithmetical geometrical relation (9/4) (Figure 6b). The second part will be considered by the Dominican friar as a philosophical, geometrical and religious defence against the considerations on the figure of Kepler's heptagon [14]. He ends the work by pointing out that, although Kepler is already condemned in the Expurgatory of the General Inquisition of the Hispanic index of 1640, the *Harmonices mundi* was not [33] and, therefore, it should also be condemned, alluding to Psalm 41, *Abyssus abysum invocat*, (one mistake calls for another). Especially damnable are the propositions 45 and 47 of the first book concerning the figures of heptagon and nonagon. The German astronomer already appeared in the general index of the *Novus Index Librorum Prohibitorum et Expurgatorum* (1632), as author [+Ioannes Keplerus] and classified *Append. Libr. Proh& Exp. I Class*. The index of his work, *Harmonices mundo*, appears only with the caveat of the dedication to James VI King of Scotland and I King of England (1566-1625) and he considers him, *Rex inter Reges*, *Fidei Defensor inter Cristi fideles* (The king, between the kings, Defender of the Faith of Christ among the faithful) [34].

The principle of this heresy is based on the assumption that Kepler claimed that the eternal Wisdom of God cannot have science for the figure of the heptagon because this figure lacks of scientific knowledge. As a consequence, he considered the cognoscibility of the figure as an *impossible simpliciter*, therefore as God did not have science, the heptagon could not have it. In order to reach this conclusion, fray Ignacio had reasoned on the axiomatic principle of the Metaphysical Schools. What it has no entity and no essence, it has no conditions, nor properties, accusing Kepler of doubting this principle. As it was impossible to inscribe the figure in the circle, based on the isosceles triangle, whose major angle is three times the minor, as fray Ignacio had demonstrated, then the

Metaphysical principle was not fulfilled in the heptagon, nonagon and in the figures of odd number and therefore they could not be considered *impossible simpliciter*.

Fray Ignacio had done so for purely mathematical reasons. He accuses Kepler of not knowing the isosceles triangle proper to the heptagon, the pentagon, as well as those of the 15-sided polygon, as Cardano and Candale had dealt with it. He refutes the non-constructability of the heptagon and the nonagon proposed by Kepler. Fray Ignacio understood that the mathematical instruments of Algebra and Geometry had to be capable of constructing any figure. Therefore it is surprising that Jost Bürgi did not do it through Algebra, or that Dürer, Candale, Carolus Marianus and the Marquis of Malaspina, had not achieved it through Geometry. It cannot be concluded, like Kepler, that the heptagon lacks cognoscibility and Science. He announced that he would soon publish the solutions of the polygons of 9, 11, 13, 15, 17 and 19 sides.

4. The construction of the regular heptagon by fray Ignacio Muñoz

In the manuscript of the *Observationes diversarum artium* (1669) it is possible to trace the reference works he may have had to acknowledge the figure of the heptagon. At the beginning of the *Geometria* (fol. 1-48) he quotes the *Geometria practica* by Cristoforo Clavio (1538-1612) in the 1606 edition in which he deals with the figure of the heptagon in Lib. VIII, Theor. 12 Propos. 30 [35], the 1604 edition princeps of Rome by Aloisio Zannetti [8].

In the chapter of the *Compendium Element: Euclidis* (fol. 433-478), done through Clavio's principles and commentaries of the *Euclidis elementorum libri XV*, in the edition at his disposal of 1589, he discusses extensively the heptagon in the *Prob. 16 Propos. 16* del LIV [36]. It is much more extensive, than in Scholion II of L. IV, f Vicent Accoltum's 1574 edition princeps of Roma, L.IV, Scholion II [37].

In the section dedicated to the *Architectura militaris* (fol. 597-626) he refers to the Latin edition of Lyon by Philippe Borde *Les fortifications du chevalier Antoine de Ville* (1640) by Antoine De Ville (1596-1657). There he solves the construction by angulation and triangulation, as he technically constructs the regular fortification [38] (Figure 8a), as well as in his edition princeps, 1629 French edition of Lyon by Irenée Barlet [39], and the *Architectura militaris moderna* (1647) by Matías Dögen (1605-1672) who develops it by angular division (Figure 8b) [40].

In Spain the methods of Juan de Arfe (1535-1603) of *De varia commensuracion para la escultura y architectura* (1585) [41] (Figure 9a) had been published. It had the same geometrical matrix as Abu'l-Wafa Al-Buzjani, (ca. 993-1008), Hans Hösch von Gmünd (1472) and Matthäus Roriczer (1488). Other methods are the construction by means of the set squarein the Andalusian tradition of Diego López Arenas (+c.1640) in the *Primera y segunda parte de las reglas de la carpintería* (1616) [42] (Figure 9b) and fray Lorenzo de San Nicolás (1593-1679) in the *Arte y Vso de Architectvra* (1633) [43]. Also we can find the angular division of the *Compendio de Arquitectura y Simetria de los Templos* (1681) by Simón García collected from Rodrigo Gil de Hontañón (1500-1577) [44] (Figure 9c), and which chronologically the Dominican friar could have known.

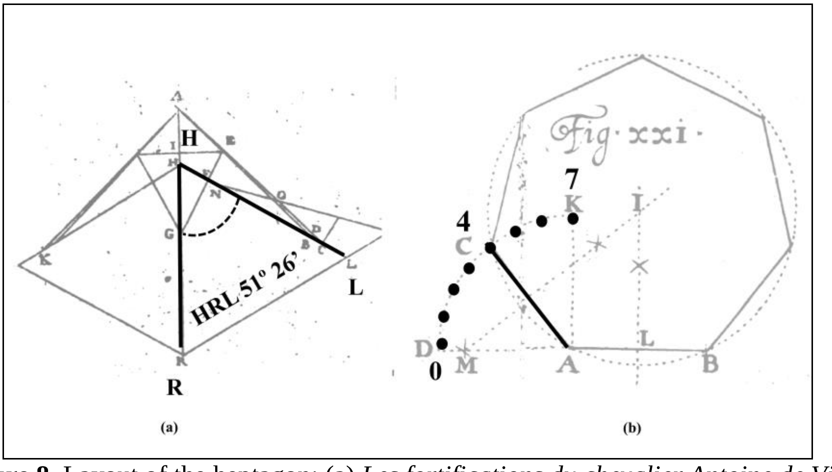

Figure 8. Layout of the heptagon: (a) *Les fortifications du chevalier Antoine de Ville* (1640) Antoine De Ville [38, Planche XXVIII, https://gallica.bnf.fr/ark:/12148/image, Gallica], (b) *Dramburgensis marchici Architectura militaris moderna* (1647), Matías Dögen [40, Fig. XXI, https://archive.org/details/bub_gb_FdUQj0ypih0C]. Interpretation by the authors. Detail: Terms of Use, Licence, Public Domain Mark.

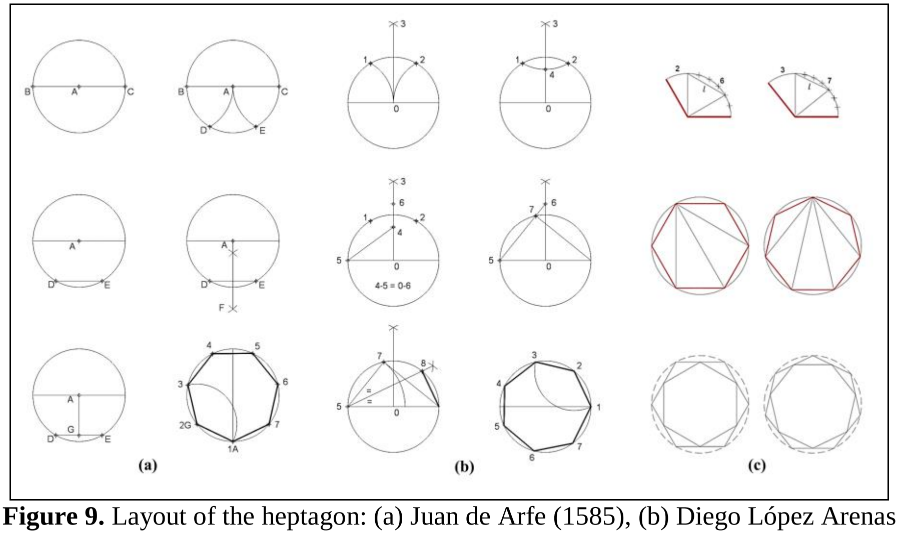

Figure 9. Layout of the heptagon: (a) Juan de Arfe (1585), (b) Diego López Arenas (1616), (c) Simón García (1681). Author's figure.

It could be speculated whether the first part of the work devoted to the construction of the heptagon is a consequence of the heretical correction of Kepler, or whether it is a consequence of his unique construction of the regular figure. According to the arrangement of the different chapters, everything suggests that Kepler's judgment is produced after having achieved, supposedly, the construction of the heptagon through a commensurable proportion with an integer base (9/4). It serves him to construct the isosceles triangle proper to the heptagon (9,4,9). The usual practice for the resolution of the heptagon was achieved from the trigonometric division as José de Zaragoza did, in the *Geometria practica Euclidis: problemata continens* (1672) [45], applied by Simón de Stevin (1548-1620) in *Lib2. pro.7* of the *Mathematicarum hypomnematun de Geometriae Praxi* (1605) [46].

Fray Ignacio approaches the heptagon's construction from the classical principle of dividing the isosceles triangle own to the regular heptagon, defined as one in which each of the two angles at the base is three times the vertical angle (Figure 6b). He reaches it through the consectary I, that triangle of double and sesquiquadratic proportion referred to its base. In a second consectary he indicates that José Zaragoza's construction is made with the trigonometric tables and therefore it is not geometric. He will add a third consectary, in which he refers the triangle (9,4,9) with the parallelogram formed by the base and 5/9 parts of the side (4,5,4,5) (Figure 6c). He prepares the Euclidean geometric base theorem to construct the heptagon, renouncing the trigonometric bases because he considers them numerical. He asserts that the proof of the theorem was to deduce a geometric impossible, such as the part and the whole being equal through the axiomatic relation (9:4) (Figure 10).

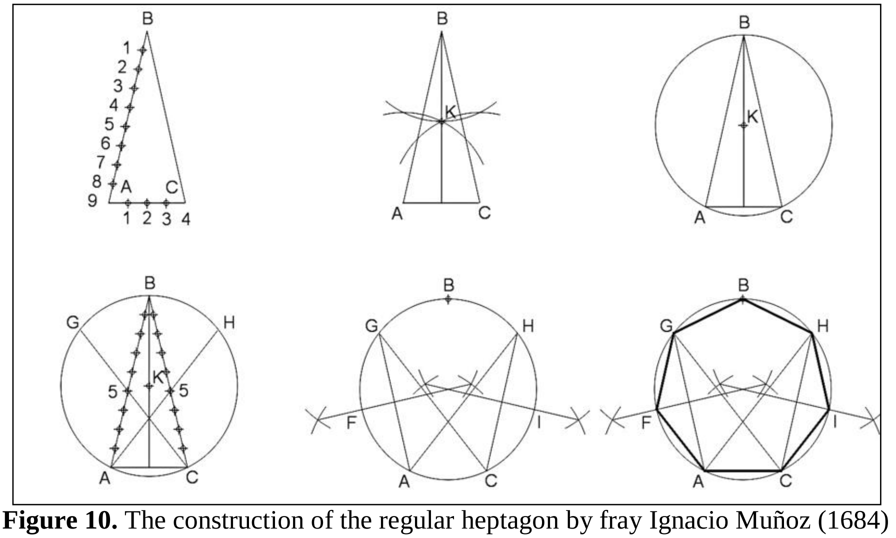

Figure 10. The construction of the regular heptagon by fray Ignacio Muñoz (1684) [14, p. 16-17]. Interpretation by the authors.

Fray Ignacio did not solve the impossible solution of the heptagon that Kepler advanced and that Carl Friedrich Gauss (1777-1855) proved in Section VII, Statements (361-366) of the *Disquisitiones Arithmeticae* (1801) [47]. He proved the impossibility of the geometrical construction of the heptagon. His axiomatic principle of the ratio (9/4), which he never revealed its origin, defined it as the geometrical impossible. It constitutes the scientific methodological genesis, and based on it, he develops the mathematical proof.

5. The 'Manifiesto geometrico' versus 'Harmonices mundi'

In the dialectic on the knowledge of the heptagon between Fray Ignacio to Kepler, a certain paradox arises. Kepler affirms that no one can consciously construct this figure and that doing so it would be within the realm of chance. At the same time, he criticizes Kepler for not having demonstrated that the figure was unconstructable, which it was true, according to Gauss's arguments. Fray Ignacio requires in the *Manifiesto Geometrico* a mathematical proof for the construction of the heptagon based on his geometrical experience from the compass. An example of this it is the chapter of the *Observationes* (fol. 770-784) in which he defines the compass of the Duke of Béjar (Figure 11a) as *planifolado universival* and describes the pantographic compass (Figure 11b) similar to the one in Clavius' *Geometria practica* [35, p. 5] (Figura 11c).

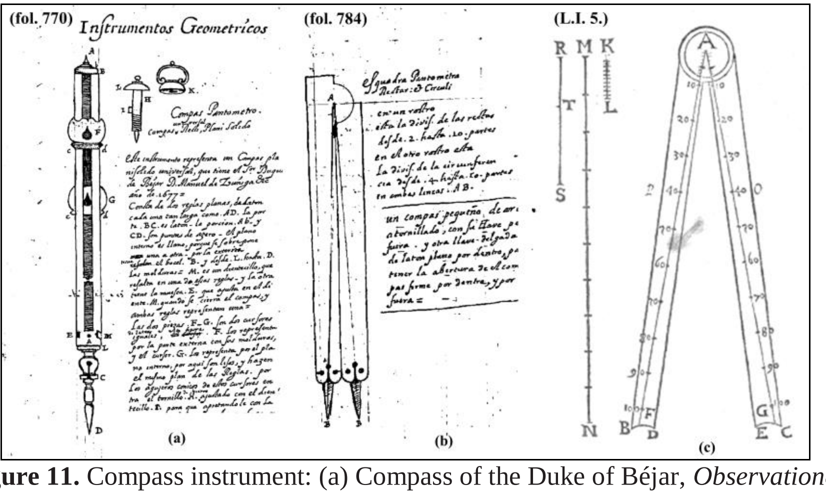

Figure 11. Compass instrument: (a) Compass of the Duke of Béjar, *Observationes diversarum artium* (1669), Ignacio Muñoz (fol.770) [BNE Mss/7111]; (b) pantographic compass, idem, (fol.784); (c) *Geometria practica* (1606), Cristoforo Clavio [35, p. 5]. Detail: Terms of Use, Licence referenced Public Domain Mark.

The construction is based on the isosceles triangle of the Kepler's heptagon (B,E,F) (9,4,9). The main diagonal (BF) (9) has points of intersection with the other two diagonals, I and K, where (BI = 4) (Figure 12a). Kepler in *Harmonices mundi* demonstrates the multiple possibilities of division of the diagonal BF that comply the proportional conditions assigned (Figure 12b).

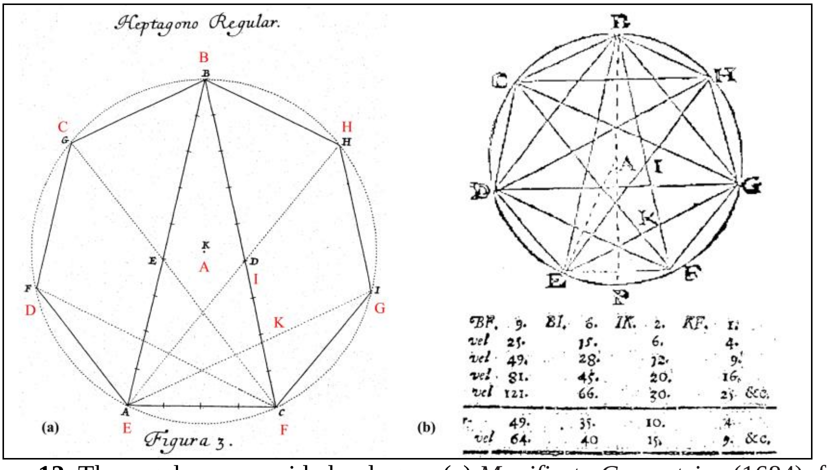

Figure 12. The regular seven-sided polygon: (a) *Manifiesto Geometrico* (1684), fray Ignacio Muñoz [14, fig. 3]; (b) *Harmonices mundi libri V* (1619), Johannes Kepler [1, p. 32-33]. Interpretation by the authors. Detail: Terms of Use, Licence, Public Domain Mark.

He concludes that no regular heptagon could have been constructed by anyone in a conscious and deliberate way through the isosceles triangle. It also cannot be achieved with this methodology either. In spite of Kepler's considerations of the unknowability of the figure for these reasons, Fray Ignacio, more than half a century later, insists on the possibility of its construction by means of an integer base proportion between the side of the heptagon (EF) of 4 units, with the diagonal of the isosceles triangle (BF) of 9 units. They both are commensurable and of finite solution. Here it begins the heretical dispute of Fray Ignacio Muñoz with Johannes Kepler, since the heptagon and the number seven represented the finite Creation (Genesis 1.1-2).

6. The background of fray Ignacio Muñoz, the 'geometia fabrorum'

The mathematician and Noyon's canon, Charles Bovelles (1478-1567) quoted Fray Ignacio in the *Observationes* (fol. 47). He is the author of the *Geometricum Introductorium* (1503) published in Paris in 1510, in which he constructs the heptagon from the hexagon of the side of the semidiameter of the circle [48] (Figure 13a). Part of the work is translated as *Geometrie en Françoy*s (1511) in which he will include a new method through the triangle formed by two contiguous sides and the diagonal between the opposite vertices [49] (Figure 13b). Years later he wrote the *Livre singulier et utile, touchant l'art praticque de geometrie* (1542). He acknowledged that a figure as important for Christian symbolism as the heptagon, as it is the number in which God creates the perfection of the world, did not appear in Euclid's *Elementa* (c.325 - c.265 BC). He defines a new layout of the heptagon by angular division (Chap. 2.60) [50] (Figure 13c), proved in the work *Geometrie practique* (1547) [51]. He deals with the decomposition of up to four types of inner isosceles triangles proper to the heptagon.

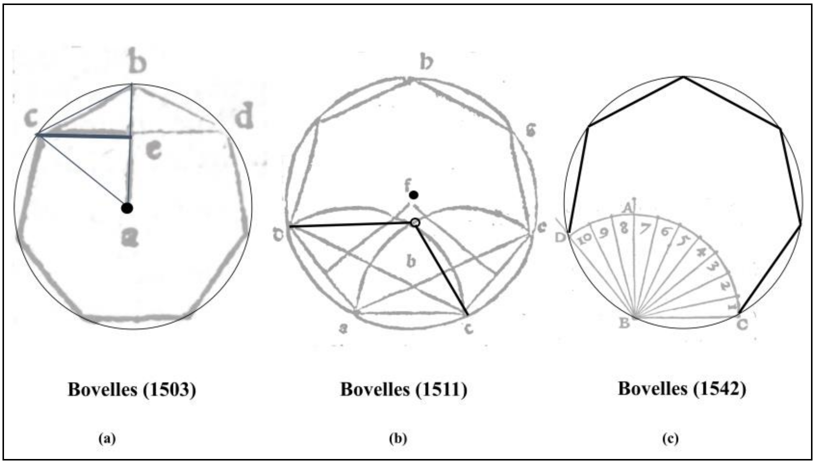

Figure 13. Drawings of the heptagon by Charles Bovelles: (a) *Libellus de Mathematicis rosis* (1510) [48, p. 196 r, https://gallica.bnf.fr/ark:/12148/bpt6k3142945/f240.image], (b) *Geometrie en Françoys* (1511) [49, p. 19r, http://www.bvh.univ-tours.fr/Consult/consult.asp?numtable=B410186201_I958 &numfiche=715&=3&offset=0&ecran=0], (c) *Livre singulier et utile, touchant l'art praticque de geometrie* (1542) [50, p. 26v, https://numelyo.bm-lyon.fr/f_view/BML:BML_00GOO0100137001100489900]. Interpretation by the authors. Detail: Terms of Use, Licence referenced Public Domain Mark.

Gothic architecture had to adapt to the new liturgy of the *Prochiron, vulgo rationale divinorum officiorum* (1291), by Guillermo de Durando (1230-1296). It resulted the creation of semi-circular ambulatories, where the different relics of saints were laid down in the radial chapels. The familiar and guild tombs could be seen without visual obstacles and orderly. This new vision replaces the allegorical one of the *Gemma animae* (c.1120) by Honorius of Autun (1080-c.1153) that the Romanesque had had with the linear relationship between the apse and the apsidioles. Then there will be a radical change in the conception of spaces, thus encouraging semi-circular apses with radial chapels, which are arranged in the form of five or seven chapels around them.

The cathedral of Noyon known to Bovelles has a polygonal apse with five radial chapels, like Burges, Reims, Sens or Tours. He knew other episcopal sees with seven chapels like Amiens, Beauvais or Chatres, which are built on the basis of a 14-sided polygon. They project out radially half side of the heptagon on the chord of the apse, a polygon which Kepler also refers to as unconstructable. The figure of the heptagon does not appear in Euclid's Elementa translated by Adelard of Bath (1075-1166) in 1142, although the pentagon (L. IV. Prop. 14) [52] and the decagon do (L. IV. Prop. 16) [52, p. 110-111]. They can be used for pentagonal apses. The heptagon does not appears in Ptolemy's *Almagesto* (c.85-165) transferred by Gerard of Cremona (1114-1187) around 1175, although the rest of the regular polygons do appear [53]. The

Gothic masters could not learn from *Elementa* and *Almagest*, which they could access directly, or through the bishops or the cathedral chapter, since there was no evidence of the figure.

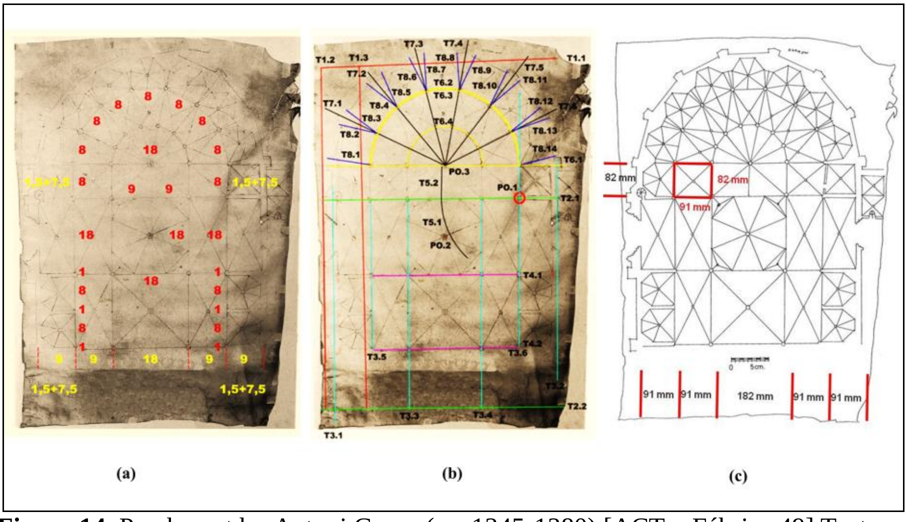

Figure 14. Parchment by Antoni Guarç (ca. 1345-1380) [ACTo, Fábrica 49] Tortosa: (a) proportion of the trace, (b) auxiliary traces of drawing lace, (c) redrawing of the parchment. Interpretation by the authors. Detail: ACTo, Fábrica 49, reproduced by kind permission of ACTo

On the other hand, evidence has been found in the design and construction of heptagonal apses from the relationship between the width of the radial chapels and the radius of the ambulatory. It shows the geometric relationship of the proportion (18/8), in other words the relationship of 9 modules between the width of the lateral nave with 8 modules and the width of the radial chapel. This is obvious in the project by Antoni Guarc (c.1345-1380), [Archivo Capitular Catedral de Tortosa (ACTo), Fabrica 49] for the cathedral of Tortosa (Figure 14a). In order to draw the seven chapels of the apse, Guarc abates the measure of the radial chapel of 8 modules on the diameter of the presbytery of module 18 (Figure 14b). He uses the numerical ratio (18/8) between the central nave and the side chapel, or in other words (9/8) between the ambulatory width and the radial chapel. Guarc's layout and the general layout of the apse built between 1383 and 1435 have the same genetic structure (Figure 14c).

Considering the relationship between the radius, the 18 modules of the semi circumference of the ambulatory, and the 8 modules of the radial chapel, a geometrical and at the same time arithmetically metrological solution is established. According to the theory of proportions; if the presbytery has a width of 18, the radial chapels must have 8 modules. In the apse, to build a chapel of 3 canas (24 palms), a radius of 6 canas and 6 palms (54 palms) is needed. So it happens in the layout of the apse of Tortosa's cathedral (Figure 15a), while the old Romanesque cathedral of 1158 pre-existed, and therefore they could not trace the circumference circumscribed to the radial chapels (Figure 16a).

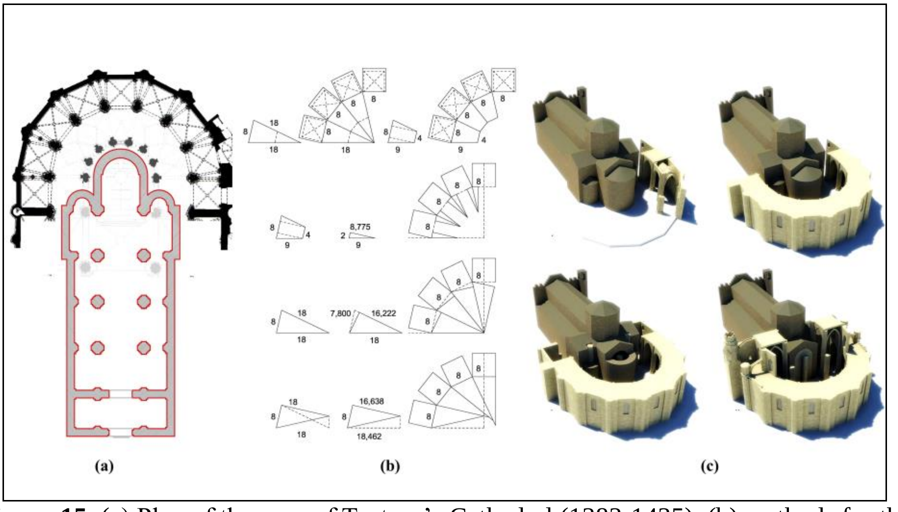

Figure 15. (a) Plan of the apse of Tortosa's Cathedral (1383-1435), (b) methods for the layout of heptagonal apses, (c) constructive evolution. Author's figure.

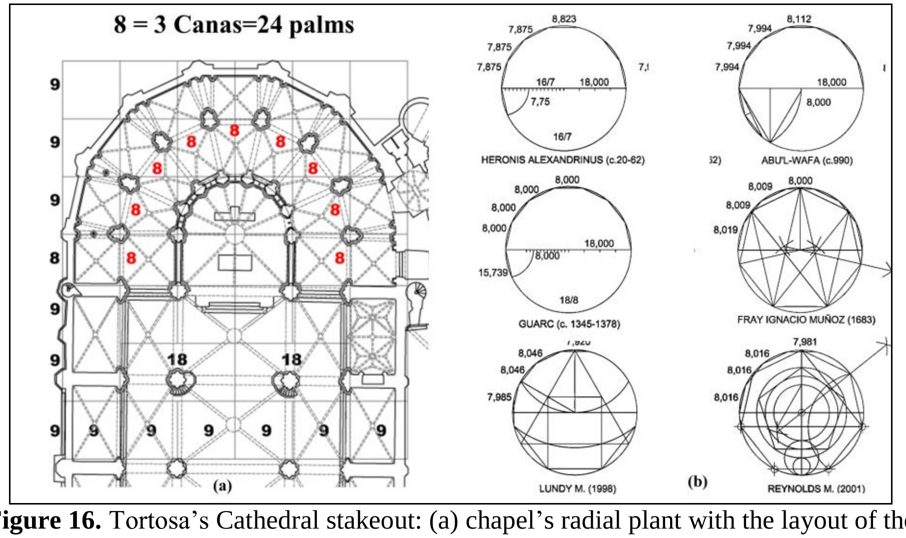

Figure 16. Tortosa's Cathedral stakeout: (a) chapel's radial plant with the layout of the Romanesque cathedral, (b) geometric proportions that allow the construction of an apse without knowing its centre, (c) construction process of the apse of the Gothic Tortosa's cathedral (1383-1435) over the Romanesque cathedral (1158). Author's figure.

Both in the Guarç parchment and in the layout of the apse, the ratio between the width of the nave (9/8) and the side chapel is used. It is the same as the ratio (18/8) between the ambulatory and the chapel. The simulation of these geometrical processes, which neither builders nor mathematicians had at their disposal, show that the results applied with this ratio (9/8) are more precise than those developed by the geometrical and mathematical treatises of the XV-XVII centuries [54].

In the Archive of Tortosa's Cathedral Chapter, some Neoplatonic codices are preserved, including that of Martianus Capella of *Nuptiis Philologiae et Mercurii* (f. 430) [ACTo 80]. It proposes two types of lines: commensurable *rhētós*, and incommensurable *álogos*. The layout of the 14-sided polygon used in the Guarç elevation, as in the execution of the Tortosa apse, uses the ratio (9/8). In the layout of the Gothic apse, the two chapels belonging to the straight section of the apse and the other seven located in the apsidal semicircle must have the same measure in their width. They have to keep the same proportion with the radius and also to have a commensurable measure of 3 canas. In terms of Martianus Capella the construction of Antoni Guarç (ca. 1345-1380) would be a line with a *rhētós* measure (Figure 17b). If it would have been used the method of the heptagon layout in the tradition of Mohammad Abu'l-Wafa Al-Buzjani, (ca. 993-100) (Figure 17a) later used in the *Geometrie Deutsch* of Hans Hösch von Gmünd (1472) and Matthäus Roriczer (1488) (Figure 17c) or those of Albert Dürer's *Underweysung der Messung* (1525) (Figure 17d-e), the construction would be *álogos*, since the mean of the line would be incommensurable. Moreover, Guarç's ratio (9/8) is understood in other cathedral codices, as in Calcidius' translation of Plato's *Timaeus* (f.350) [ACTo 80], and in Macrobius' *Comentarii In Somnium Scipionis* [ACTo 236] (f.400), as the ratio between the integer and its octave (1+1/8), which is called *epogdous*.

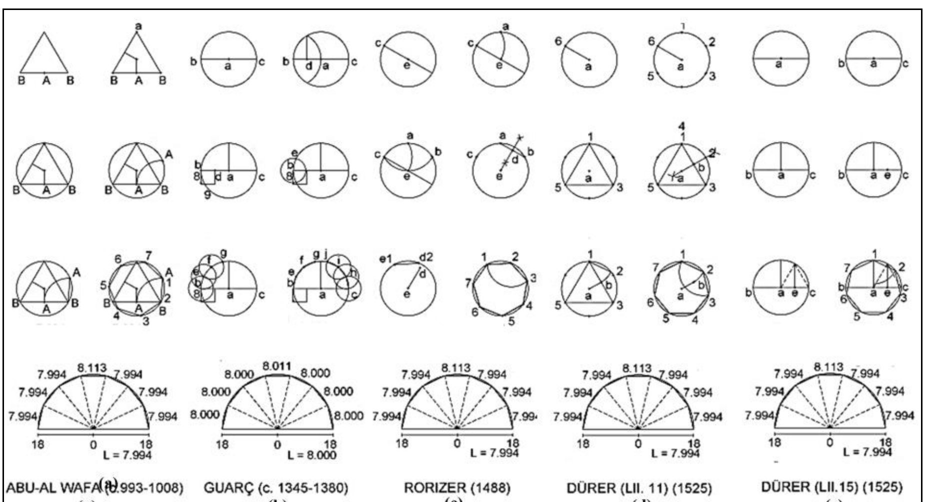

Figure 17. Application of the heptagon geometric methods in the construction of Gothic apses: (a) Abu'l-Wafa Al-Buzjani (ca. 993-1008), (b) Antoni Guarç (ca. 1345-1380), (c) Matthäus Roriczer (1488), (d) Albert Dürer LII.11 (1525), (e) Albert Dürer LII.15 (1525). Author's figure.

Therefore, the proportion used 18/8 (9/8) was well known to the developers and builders who built Tortosa's cathedral. The ratio of 18/8 arithmetically based (a/b) as well as geometrically based, is similar to the one proposed by the Dominican friar for his isosceles triangle of 9/4. Therefore the method in the *Manifiesto Geometrico* could be used to construct a heptagonal apse obtained from the 14-sided polygon (Figure 15a). It would be, among those published with geometric development, the most accurate (Figure 15b).

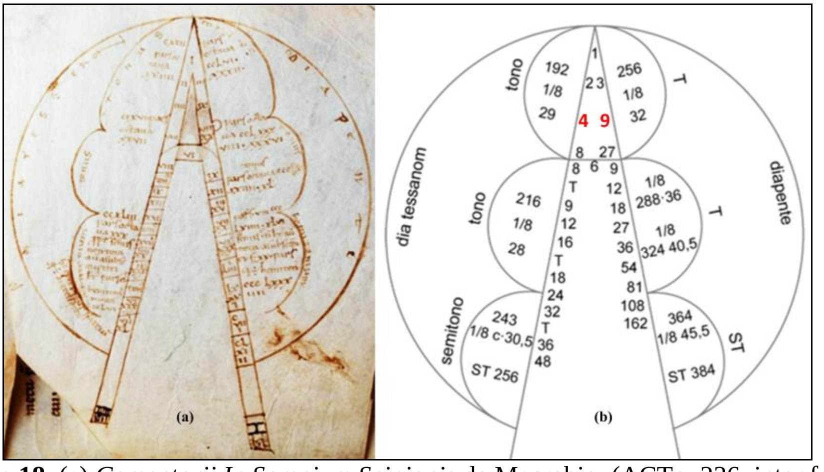

Figure 18. (a) *Comentarii In Somnium Scipionis* de Macrobio, (ACTo. 236, inter fol. 51 v-52 r); (b) interpretation by the authors, Ratio (9:4). Author's figure. Detail: (ACTo. 236) reproduced by kind permission of ACTo).

The measurements are related to numerical modulations, the *diapente* (3:2), the *diatessaron* (3:4) and the tone (9:8) well known to the cathedral canons who had read in the [ATCo 80] and the [ACTo 236] (Figure 18), and similar to the (9:4) of fray Ignacio Muñoz. Neither the method nor the proportion of Antoni Guarç (ca. 1345-1380) [ACTo, Fabrica 49] appears in scholarly treatises, but it is an instrument of the *geometria fabrorum* that provides a solution that is simultaneously geometrical and arithmetical. In the Tortosa's cathedral, the measurement of the chapel which it measures 3 canas in length (24 palms) and all the measurements of the apse, both in plan and in section, are implemented as an algorithm.

Fray Ignacio Muñoz proposes the third problem as a conclusion of the first part of the *Manifiesto Geometrico* [14, p. 19]. How to construct a regular heptagon without the circle that circumscribes it, which there is no solution. The question is the same as the Gothic builders had in the design of their apsesbecause in many cases the Gothic cathedrals replaced the Romanesque churches. This happened as the primitive presbyteries continued to function while the new cathedral apse was being built. The geometrical solution is achieved by using some triangles such as those of the Dominican friar, (9, 8+3/4, 2), (18,8,18) and known trapezoids (8,9,4,9) [55] (Figure 16b). They allow the construction of the radial chapels without determining the circumscribed circumference (Figure 16c). These figures of Guarc's metrology of base (18/8) or Fray Ignacio's (9/4) are similar to those of Liber II. Prop.77 of the Hibbur ha-Meshihah ve-ha-Tishboret (1116) by Abraam Bar Hiia (1070-1136) [56] and the *Practica geometriae's* (1223) by Leonardo Pisano (c.1180-1250)

These geometric methods to trace the figure without knowing its centre based on triangulation, could also be solved from the external or inner angle of the heptagon. This tradition starts from the French military architecture [57] and known by fray Ignacio Muñoz through the authors cited in the *Observationes diversarum artium* (1669), such as Matías Dögen in the *Architectura militaris moderna* (1647). He simplifies the angular division by means of the proportion (7/4) for its external angle (Figure 8b), very similar to the one used by Charles Bovelles (1547) with the proportion (10/7), but in this case with inner angle (Figure 13c). Indeed he specifies that nobody had solved geometrically the division of the right angle in seven parts.

He also knows the solution of the Latin edition of 1640 by Antoine De Ville, in *Les fortifications du chevalier Antoine de Ville* [38] (Figure 8a) and the solution for any polygon already published in the first French edition [39, p. 38]. These methodologies of heptagon layouts come from Le *Gouvernail d'Ambroise Bachot* (1598) by Ambroise Bachot (d.1587) [58] in which the approximation of the angle between its sides is (128+1/2)º (Figure 19a). In *La fortification reduicte en art* (1600) by Jean Errard (1554-1610) [59] the measure of the central angle is (51+3/7)º. Both of them are used to determine the walls and flanks of defensive fortifications.

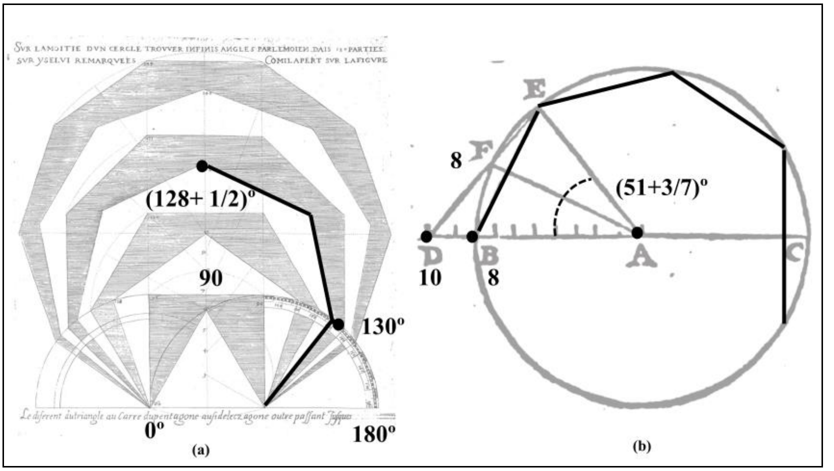

Figure 19. The construction of the heptagon in military engineering: (a) *Le Gouvernail* (1598), Ambroise Bachot, general solution of the construction of regular polygons [58, s.n., https://gallica.bnf.fr/ark:/12148/bpt6k15139955]; (b) *La fortification reduicte en art* (1600), Jean Errard, construction of the heptagon [59, p. 22r-23, https://gallica.bnf.fr/ark:/12148/bpt6k85639h.image]. Interpretation by the authors. Detail: Terms of Use, Licence referenced Public Domain Mark.

Therefore the Dominican friar must not have known the principle of the Gothic masters for tracing the heptagonal apses without knowing their centre. He also ignored the property of the precision of the proportion (9:4) between the diameter of the circumference and the 14-sided polygon for its construction.

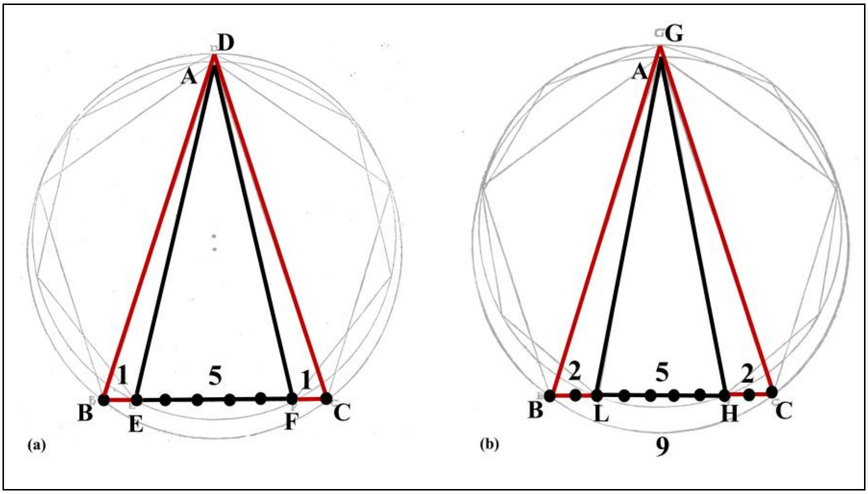

Figure 20. Geometric constructions with the pentagonal base *Der furnembsten, notwendigsten, der gantzen Architectur angehörigen mathematischen vnd mechanischen Künst eygentlicher Bericht* (1547), Walther Hermann Ryff, [https://archive.org/details/derfurnembstenno00ryff]: (a) construction of the heptagon of side 5 through the pentagon of side 7 [60, p. XXX], (b) construction of the enneagon of side 5 through the pentagon of side 9 [60, p. XXXI]. Interpretation by the authors. Detail: Terms of Use, Licence referenced Public Domain Mark.

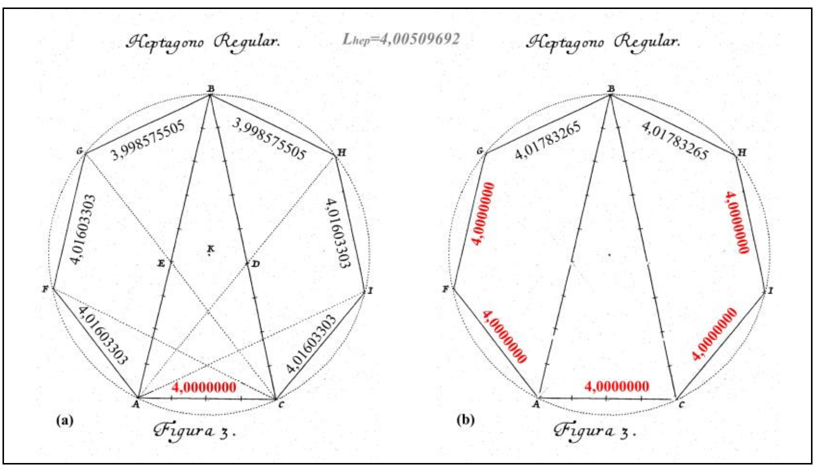

Figure 21. (a) Metric solution of the construction of the heptagon of fray Ignacio Muñoz [14, Fig. 3], (b) alternative solution to the construction of fray Ignacio Muñoz [14, Fig. 3]. Interpretation by the authors. Detail: Terms of Use, Licence referenced Public Domain Mark.

He doesn't know the thesis of Walther Hermann Ryff (c.1500-1548) of the, *Der furnembsten, notwendigsten, der gantzen Architectur angehörigen mathematischen vnd mechanischen Künst eygentlicher Bericht* (1547) in which he theorizes about the construction of the polygons (5, 7, 9, 11, 13...). He starts from the isosceles triangle of integer base in which he creates a relationship between the pentagon of commensurable construction with other polygons. The heptagon is constructed through the isosceles triangle of base 5 and sides DE and DF which are equivalent to the equal sides of the pentagon AB and AC of side 7 and whose measure would be ABC (11.27568241, 7, 11.27568241). The solution to the isosceles triangle of the heptagon DEF is (11.23489801, 5, 11.23489801). It is observed that the measures of the diagonals of the figures are not equal (Figure 20a). In Fray Ignacio’s solution the heptagon would have been (11.25, 5, 11.25). He uses the same methodology for the construction of the nine-sided polygon, whose base is also 5. He uses the sides of the isosceles triangle of the pentagon of side 9, ABC (14.56230590, 9, 14.56230590) and whose solution for that of the enneagon GHL would be (14.39692621, 5, 14.39692621) (Figure 20b) [60].

Pietro Cataneo (d.1569) in *L’ architettura* (1567) deals with the construction of the heptagon and the generalized solution of odd polygons with a system similar to Ryff’s. He divides the side of the circumscribed equilateral triangle in as many parts as sides we want to divide the circumference. He takes these three units as the side of the polygon we want to construct [61].

In the solution of Fray Ignacio the sides of the heptagon are not equal, having three different results; [4.0000000, 4.01603303, 4.01603303, 4.01603303, 4.01603303, 4.01603303, 3.998575505, 3.998575505]. The cause of this configuration is his false mathematical basis of the isosceles triangle and therefore the relationship of the side (EF = 4) of 4 units. The approximation of the true magnitude of the major angle of the isosceles triangle is [77.14285714º], as opposed to the one deduced by Fray Ignacio [77.16041159º] (Figure 21a).

If he had not this statement of the construction through the isosceles triangle proper to the heptagon, which Kepler had refuted as non-constructable, he could have stated: ‘In the circumscribed circumference of the isosceles triangle of the heptagon (9, 4, 9), the side of the heptagon is equal to the base of this triangle (4)’. With this new statement, five of the sides would have (4 u) [4.0000000, 4.0000000, 4.0000000, 4.0000000, 4.0000000, 4.0000000, 4.01783265, 4.01783265], with a computer approximation of the side of the heptagon of [4.00509692], and therefore more precise than the one he proposes (Figure 21b).

The mathematical principle of the division of the heptagon on the basis of the isosceles triangle (9,4,9) and the numerical inequality of the different results of the measurement of its sides, which Fray Ignacio could not prove, will be discussed by the King’s engineer Jorge del Pozo (d. 1676). He had occupied the *Chair of Mathematics, Fortification and Artillery* (1667-1678), dependent on the Council of War and he wrote the booklet *Responde Jorge del Poço desde la otra vida, como catedratico que fue de mathematicas en la Chanuerga, al papel impresso en Bruxelas este presente año de 1684 sacado a la luz por el padre maestro Fray Ignacio Muñoz* [M-RAH, 9/2782]. Despite this, Niccolò Coppola (+ 1697), does not mention the work of Fray Ignacio in the *Formacion exacta del heptagono: geometricamente hallada por medio de la linea commensuratriz del quadrante* in which he discusses the trisection of the angle that had been published in Mateo Fernández de Rozas (+1697) that same year in Madrid as the *Geometric solution of the famous angle trisection problem* [62].

7. Conclusions

The first part of the *Manifiesto Geometrico* (1684) by Fray Ignacio Muñoz wants to demonstrate mathematically the construction of the heptagon by means of a deductive methodology. It is based on an axiomatic principle of the isosceles triangle of the heptagon (9,4,9) which is erroneous, but which he argues rigorously. The axiomatic principle of the proportion (9/4) on which he bases the construction of the polygon, and which he never reveals, could be interpreted in different ways. From the arithmetical point of view as the relation 9/4 = 4/4+4/4+2/8 = 2+1/4, from the geometrical as the relation of a double square and its fourth, from the musical proportion as the double plus a sesquiquiquadrate and from the astronomical in the reference to the distance to the Sun between Jupiter 9 and Venus 4.

The second part becomes an apology against Johannes Kepler, who had argued in *Harmonices mundi libri V* (1619) the unknowability of the heptagon as an infinite figure, and hence the argumentation of the heretical principle of the German astronomer. In Genesis the Creation is finite, in seven days, and in the vision of the Kepler's indeterminate and unconstructable, hence the Dominican's inquisitive outburst. On the other hand, the modulation (9:4), as well as the construction of the heptagon without inscribing it in a circle, proposed in the *Manifiesto Geometrico*, was known in the *geometria fabrorum*. It was used for the construction of the apses of the Gothic cathedrals with seven radial chapels whose bases are in the neoplatonic transmission of the proportion of numerical whole base. Therefore, commensurable of Chalcidius, Capella and Macrobius that the developers of these constructions knew well, and that Fray Ignacio must not have known.

Fray Ignacio Muñoz died without knowing that his method for the construction of the heptagon is one of the most precise methods of practical geometry that has been developed to date. This fulfils Kepler's prediction that either it cannot be constructed, or that if it is constructed, it is not possible to verify it.

The geometrical contributions of Fray Ignacio Muñoz did not advance the mathematical science of the 17th century. Nevertheless, his good knowledge helped him in the practical resolution of the problems that a missionary needed for the work of evangelization. Especially for the dogmatic safeguarding of the Catholic faith, carried out by the reference of the *Index Librorum Prohibitorum et Expurgatorum*, in which the 17th century Inquisition had put on trial astronomers such as Copernicus, Galileo or Giordano Bruno.

The *Manifiesto Geometrico* can be understood as a palimpsest of the geometric tradition for the construction of the heptagon in the seventeenth century. A superimposed knowledge, although unknown to the author, in the *geometria fabrorum* of the builders of the Gothic apses. The *magister operis*, also probably without knowing it, are the *neusis* constructions of the heptagon of Greek geometry.

Acknowledgement

To Architect Valentina Risemberg, for proofreading the text.

References

[1] J. Kepler, *Ioannis Keppleri Harmonices mundi libri V.: quorum primus Geometricus, de figurarum regularium, quae proportiones harmonicas constituunt, ortu & demonstrationibus,* sumptibus Godofredi Tampachii bibl. Francof.: excudebat Ioannes Plancus, Lincii Austriae, 1619, 32-40.
[2] M. Caspar, *Johannes Kepler Gesammelte Werke Band XV. Briefe 1604-1607. Herausgegeben Von Max Caspar,* C.H. Beck'sche Verlagsbuchhandlung, Munchen, 1951, 145-147.
[3] M. Caspar, *Johannes Kepler Gesammelte Werke Band VI. Harmonice Mundi. Herausgegeben Von Max Caspar,* C.H. Beck'sche Verlagsbuchhandlung, Munchen, 1940, 47-56.
[4] J. Kepler, *Prodromus dissertationum cosmographicarum, continens mysterium cosmographicum, de admirabili proporione orbium coelestium, deque causis coelorum numeri, magnitudinis, motuumque periodicorum genuinis & propriis, demostratum, per quinque regularia corpora geometrica,* Excudebat Georgius Gruppenbachius, Tubinga, 1596, 24-25.
[5] W.R. Knorr, Centaurus, **32(4)** (1998) 257-271.
[6] R. Rashed, Journal Historical Arabic Science, **3(2)** (1976) 387-409.
[7] J.P. Hogendijk, Arch. Hist. Exact Sci., **30(3-4)** (1984) 197-330.
[8] C. Clavio, *Christophori Clavii Bambergensis e Societate Iesu, Geometria practica*, ex typographia Aloisij Zannetti, Romae, 1604, 407-409.
[9] J. Aiton, A.M. Duncan, J.V. Field, *The Harmony of the World by Johannes Kepler*, The American Philosophical Society, Philadelphia, 1997, 60-79.
[10] G. Cardano, *Hieronymi Cardani medici mediolanensis De subtilitate libri XXI*, apud Ioh. Petreium, Norimbergae, 1550, 306.
[11] G. Cardano, *Hieronymus Cardano, Mediolanensis, Medici. Subtilitate Libri XXI*, Apud Gulielmum Rouillium, Lugduni, 1559, 546-549.
[12] D.E. Rowe, A. Heeffer and T. Rothman, Math. Intell., **36(4)** (2014) 53-66.
[13] C.A. Cardona Suárez, *La geometría de Alberto Durero: estudio y modelación de sus construcciones*, Fundación Universidad de Bogotá Jorge Tadeo Lozano, Bogotá, 2006, 173-177.
[14] I. Muñoz, *Manifiesto geometrico, plus ultra de la geometria practica: addicion al IV libro de los Elementos de Euclides: construccion y demostracion geometrica del triangulo isosceles propio del heptagono regular y descripcion de la misma figura*, Francisco Foppens, Bruselas, 1684, 20-40.
[15] J.A. Sánchez Pérez, *La Matemática La matemática española en el siglo XVII*, Gráfica Universal, Madrid 1935, 10-11.
[16] A. Durer, *Underweysung der Messung, mit dem Zirckel und Richtscheyt: in Linien Ebnen vo gantzen Corporen*, Hieronymus Andreae, Nüremberg, 1525, 27v-28v.
[17] G. Bibiena, *L'architettura civile, preparata su la geometria e ridotta alle prospettive*, P. Monti, Parma, 1711, 10.

[18] M.C. Ghyka, *Estética de las proporciones en la naturaleza y en las artes*, Poseidón, Barcelona, 1983, 79.
[19] M.F. Woepcke, Journal asiatique, **V(Février-Avril)** (1855) 218-359.
[20] D. Raynaud, Hist. Math., **39(1)** (2012) 34-83.
[21] C.A. Heideloff, *Die Bauhütte des Mittelalters in Deutschland*, Johan Adam Stein, Nürnberg, 1844, 96-97.
[22] M. Roriczer, *Das Büchlein von der Fialen Gerechtigheit (fak. Regensburg 1486) und Die Geometria Deutsch*, Hürtgenwald Guido Pressler, Regensburg, 1999, 56-60.
[23] M. Curtze, Mitteilungen des Coppernicus-Vereins zu Thorn, **6** (1887) 25-32.
[24] F. Viète, *Francisci Vietae Opera Mathematica. In unum Volumen congesta ac recognita*, ExOfficina Bonaventurae & Abrahami Elzeviriorum, Lugduni Batavorum, 1696, 362-363.
[25] H. Schoene, *Heronis Alexandrini opera quae supersunt omnia*, Vol. III: *Rationes dimetiendiet commentatio dioptrica*, Teubner, Leipzig, 1903, 55-59.
[26] G. Friedlein, *De institutione arithmetica libri duo. De institutione Musica libri quinque*, in *Accedit geometria quae fertur Boetii*, B.G. Aedibus (ed.), Teubneri, Leipzig, 1867, 420-421.
[27] G. Valla, *De expetendis et fugiendis rebus*, Impr. Aldo Manuce, Venetiis, 1501, sn Exp.et Fug. Lib XIIII et Geometriae V.
[28] J.M. Moreno, Anu. Estud. Am., **78(1)** (2021) 45-78.
[29] M.L. Rodríguez-Sala, *Fray Diego Rodríguez: Astrónomo-Astrólogo-Matemático, precursor de la modernidad científica nacional*, in *Del estamento ocupacional a la comunidad científica: astrónomos-astrólogos e ingenieros (siglos XVII al XIX)*, M.L. Rodríguez Sala (ed.), Universidad Nacional Autónoma de México, Ciudad de México, 2004, 85-130.
[30] V. Navarro, Cronos, **2(1)** (1999) 105-144.
[31] J.M. González, *Historia de las Misiones Dominicanas de China*, vol. 5, Imprenta de Juan Bravo, Madrid, 1967, 407.
[32] D. Gomà, Scripta Nova, **XVI(418)** (2012) 19, online at http://www.ub.es/geocrit /sn/sn-418/sn-418-19.htm.
[33] A. Soto Mayor, *Novissimus librorum prohibitorum et expurgandorum index pro catholicis Hispaniarum regnis, Philippi IIII. Reg. Cath. ann. 1640 Iussu ac studijs D. Antonij a Soto Maior*, ex Typographaeo Didaci Diaz, Madriti, 1640, 627-629.
[34] A. Zapata, F. Lyra, J. Herrera and A. Popma, *Novus Index Librorum Prohibitorum et Expurgatorum*: Ex Typographeo Francisci de Lyra, Hispali, 1632, 595-596.
[35] C. Clavio, *Cristophori Clavii Bambergenensis Societate Iesu. Geometria practica*, Typographeo Ioannis Albini, Moguntia, 1606, 362-364.
[36] C. Clavio, *Elementorum libri XV; accessit XVI de solidorû regularium cuiuslibet intra quodlibet comparatione: omnes perspicuis demonstrationibus, accuratisque scholijs illustrate*, apud Bartholomaeum Grassium, apud Sanctium, & soc, Romae, 1589, 494-495.
[37] C. Clavio, *Euclidis Elementorum libri XV: accessit XVI de solidorum regularium comparatione ... auctore Christophoro Clauio Bambergensi*, apud Vincentium Accoltum, Romae 1574, 142r-143r.
[38] A. De Ville, *Les fortifications du chevalier Antoine de Ville, contenans la manière de fortifier toute sorte de places... comme aussi les ponts, passages, entrées de rivieres, ports de mer... la construction de toutes sortes de forts et citadelles... Avec l'attaque, et les moyens de prendre les places... Pauis la défense... Le tout représenté en cinquante-cinq planches*, chez Philippe Borde, Lyon, 1640, 29-30.

[39] A. De Ville, *Les fortifications du chevalier Antoine de Ville, contenans la manière de fortifier toute sorte de places... comme aussi les ponts, passages, entrées de rivieres, ports de mer... la construction de toutes sortes de forts et citadelles... Avec l'attaque, et les moyens de prendre les places... Pauis la défense... Le tout représenté en cinquante-cinq planches*, Irenée. Barlet, Lyon, 1629, 29-30.

[40] M. Dögen, *Dramburgensis marchici Architectura militaris moderna: varijs historijs, tam veteribu Matthiae Dögen Dramburgensis marchici Architectura militaris moderna: varijs historijs, tam veteribus quam novis confirmata, et praecipuis totius Europae munimentis, ad exemplum adductis exornata*, Apud Ludovicum Elzevir, Amstelodamiium, 1647, 26-27.

[41] J. Arfe, *De varia commensuracion para la escultura y architectura*, Andrea Pescioni y Juan de León, Sevilla, 1585, 7v.

[42] D. López de Arenas, *Breve Compendio de la Carpintería de lo blanco y tratado de alarifes*, Luis Estupiñan, Sevilla, 1633, 16-17.

[43] L. de San Nicolás, *Arte y uso de la Arquitectura. Dirigida Al Patriarca S. Ioseph. Compuesto por Fr. Laurencio de S Nicolas, Agustino Descalço, Maestro de obras*, s.n, Madrid, 1633, 80v.

[44] S. García, *Compendio de arquitectura y simetría de los templos conforme a la medida de cuerpo humano con algunas demostraciones de geometría: año de 1681*, Colegio Oficial Arquitectos Valladolid, Valladolid, 1991, l64.

[45] J. de Zaragoza, *Geometria practica Euclidis: problemata continens*, apud Bernardum à Villa-Diego, Matriti, 1672, 69-71.

[46] S. Stevin, *Tomus Secundus Mathematicarum hypomnematun de Geometriae Praxi*, Ex Officina Ioannis Patii, Academiae Typographi, Lugodini Batavorum, 1605, 66.

[47] C.F. Gauss, *Disquisitiones Arithmeticae. Auctore D. Carolo Federico Gauss*, in *Commissis apud Gerh*, Fleischer Jun, Lipsiae, 1801, 454-463.

[48] C. Bovelles, *Liber de intellectu, Liber de sensibus, Libellus de Nihilo, Ars oppositorum, Liber de generatione. Liber de Sapiente, Liber de duodecim numeris*, Henri Estienne, Paris, 1510, 196.

[49] C. Bovelles, *Geometrie en Françoys. Novellement Imprime a Paris par Henri Estienne Imprimeur et libraire demourant en la rue saint Jehan de beauvoys*, Henri Estienne, Paris, 1511, 18v-19v.

[50] C. Bovelles, *Livre singulier et utile, touchant l'art praticque de geometrie, compose nouvellement en françoys, par maistre Charles de Bouvelles*, Regnaud Chaudière et Claude, Paris, 1542, 25v-28r.

[51] C. Bovelles, *Geometrie practique*, De l'imprimerie de Regnaud Chaudiere, & Claude son filz, Paris, 1547, 25v-28r.

[52] T.L. Heath, *The thirteen books of Euclid's elements V2*, University Press, Cambridge, 1908, 106-107.

[53] G.F. Toomer, *Ptolemy's Almagest. Translated and annotated by G.J. Toomer. With a forewordby Owen Gingerich*, Duckworth, London, 1984, 35-74.

[54] J. Lluis i Ginovart, M. López-Piquer and J. Urbano-Lorente, Nexus Netw. J., **20(2)** (2018) 153-172.

[55] J. Lluis i Ginovart, G. Fortuny Anguera, A. Costa Jover and P. de Sola-Morales Serra, Nexus Netw. J., **15(2)** (2013) 325-348.

[56] J.M. Millàs Vallicrosa, *Llibre de geometria: Hibbur hameixihà uehatixbòret*, Alpha, Barcelona, 1931, 62.

[57] D. Raynaud, *La construction des polygones réguliers au xviie siècle: ingénieurs et mathématiciens*, in *Géométrie pratique, ingénieurs et architectes XVI*^{e}*-XVIII*^{e} *siècle*, D. Raynaud (ed.), Presses universitaires de Franche-Comté, Besançon, 2015, 61-81.
[58] A. Bachot, *Le Gouvernail d'Ambroise Bachot, capitaine ingenieur du Roy*, Imprime l'auteur soubz; et aussi s'en trouuera Son en Logis a` Paris, Melun, 1598, s.n.
[59] J. Errard, *La fortification reduicte en art par Errad de Bar-Le-Duc*, s.n. Paris, 1600, 22r-23v.
[60] W.H. Ryff, *Der furnembsten, notwendigsten, der gantzen Architectur angehörigen mathematischen vnd mechanischen Künst eygentlicher Bericht: vnd vast klare, verstendliche Vnterrichtung, zu rechtem Verstandt der Lehr Vitruuij, in drey furneme Bücher abgetheilet: als, Der newen Perspectiua ...: der Geometrischen Büxenmeisterey, vnd Geometrischen Messung*, Johann Petreius, Nürnberg, 1547, XXIX r-XXXr.
[61] P. Cataneo, *L' architettura di Pietro Cataneo senese. Alla quale oltre all'essere stati dall'istesso autore riuisti, meglio ordinati e di diuersi disegni, e discorsi arricchiti i primi quattro libri per l'adietro stampati, sono aggiunti di piu il quinto, sesto, settimo e ottauo libro*, Aldus, Venice, 1567, 154-155.
[62] N. Coppola, *Formacion exacta del heptagono: geometricamente hallada por medio de la linea commensuratriz del quadrante por el doctor A. Nicolas Coppola; repitese la resuelta y demostrada triseccion del angulo, inventada por el mismo Doctor Coppòla respondiendo à ... D. Matheo Fernandez de Rozas*, [s.n.], Madrid, 1693, 1-10.